deformations.

The author is grateful to V.L. Berdichevskii for discussing the results.

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DYNAMIC DEFORMATION OF INCOMPRESSIBLE MEDIA*

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A class of plane and axisymmetric problems concerning incompressible media with power law hardening, deformed over time according to special laws, is considered. Such media include, in fact, hardening plastic, non-linearly elastic and non-linearly viscous bodies whose compressibility can be neglected. The dynamic effects are studied under which the points of the body execute oscillatory or monotonic motions with respect to time. The external forces corresponding to dynamic deformation of the media in question are given. Problems of unloading are omitted for brevity; only the stages of the motion leading to loading will be considered.

Wave processes in plastic and other non-linear compressible bodies have been investigated in many papers (/1-8/ et al.). The problems of dynamic deformation under the assumption that the material is incompressible merits special attention, especially from the point of view of determining how the inertial forces affect the strength of the bodies.

1. Plane deformation. The relations for the medium in question under the conditions of plane deformation are given in polar coordinates and in the usual notation in the form of: the equations of motion

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}=\rho\frac{\partial^2u}{\partial t^2} \tag{1.1}$$
$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2}{r}\tau_{r\theta}=\rho\frac{\partial^2v}{\partial t^2}$$

the relation connecting the deformation and stress intensities, and the relations connecting the deformation, stress and displalement components

$$\varepsilon_0=k\sigma_0^{\ n} \tag{1.2}$$
$$\varepsilon_0=\sqrt{(\varepsilon_r-\varepsilon_\theta)^2+4\gamma_{r\theta}^2},\quad \sigma_0=\frac{1}{2}\sqrt{(\sigma_r-\sigma_\theta)^2+4\tau_{r\theta}^2}$$

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$$\varepsilon_{ij} = {}^1/_2 k\sigma_0^{n-1}(\sigma_{ij} - \delta_{ij}\sigma), \quad \varepsilon_r = \partial u/\partial r$$
$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad 2\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}$$

We will seek the stresses and displacements in the form (a prime denotes a differentiation with respect to θ, and a dot denotes differentiation with respect to time

$$\begin{aligned}
&\sigma_r, \sigma_\theta = \varkappa n\lambda^{-1} f^{1/n} r^{-\lambda/n} \{([\psi'' - \lambda(\lambda-2)\psi]\chi)' + \\
&\quad \nu\psi' + 2(\lambda-1)(\lambda/n - 2 \mp \lambda/n)\psi'\chi\} - \\
&\quad \rho M^{\cdot\cdot} r\sin\theta + \rho N^{\cdot\cdot} r\cos\theta + H, \quad \varkappa = \pm 1 \\
&\tau_{r\theta} = \varkappa f^{1/n} r^{-\lambda/n}[\psi'' - \lambda(\lambda-2)\psi]\chi, \quad \nu = \pm\rho k\mu^2 \\
&\chi = \{[\psi'' - \lambda(\lambda-2)\psi]^2 + 4(\lambda-1)^2\psi'^2\}^{(1-n)/(2n)} \\
&u = \varkappa k f r^{1-\lambda}\psi' - M\sin\theta + N\cos\theta \\
&v = \varkappa k(\lambda-2) f r^{1-\lambda}\psi - M\cos\theta - N\sin\theta, \ \lambda \neq 0
\end{aligned} \tag{1.3}$$

Here f, M, N, H are arbitrary functions of t, ψ is an arbitrary function of θ, and μ and λ are constants. Here and henceforth the parameters λ and n are independent when $\mu = 0$, and we have $\lambda = 2n/(n-1)$ when $\mu \neq 0$.

The expressions for the stresses and displacements (1.3) given above will represent a solution of the system of Eqs. (1.1), (1.2), provided that $\psi(\theta)$ satisfies the following fourth-degree equation:

$$\begin{aligned}
&([\psi'' - \lambda(\lambda-2)\psi]\chi)'' + \lambda n^{-1}(2 - \lambda n^{-1})[\psi'' - \lambda(\lambda-2)\psi]\chi + \\
&4(\lambda-1)(\lambda n^{-1} - 1)(\psi'\chi)' + \lambda\nu n^{-1}(\lambda-2)^{-1}[\psi'' + \\
&(\lambda-2)^2\psi] = 0
\end{aligned} \tag{1.4}$$

and $f(t)$ the second-degree equation

$$f^{\cdot\cdot} \pm \mu^2 f^{1/n} = 0 \tag{1.5}$$

whose solution is given, when $\mu \neq 0$, in quadratures by

$$\sqrt{\frac{2}{m}}\,\mu t = \pm\int_{f_0}^{f}\frac{dx}{\sqrt{c^m \mp x^m}}, \quad m = 1 + \frac{1}{n} \tag{1.6}$$

where f_0, c, μ are parameters characterizing the dynamic deformation. The minus sign in the radicand corresponds to the non-linear oscillation of the body ($n = 1$ indicates an harmonic oscillation), and the plus sign corresponds to a deformation monotonic in time.

When $\lambda = 0$, the system (1.1)-(1.2) has the solutions

$$\begin{aligned}
&\sigma_r, \sigma_\theta = (\rho r^2 h^{\cdot\cdot} \pm \varkappa h^{1/n})\cos 2(\theta - \delta) - \rho M^{\cdot\cdot} r\sin\theta + \\
&\quad \rho N^{\cdot\cdot} r\cos\theta + H, \quad \tau_{r\theta} = -\varkappa h^{1/n}\sin 2(\theta - \delta) \\
&u = {}^1/_2\varkappa k h r\cos 2(\theta - \delta) - M\sin\theta + N\cos\theta \\
&v = -{}^1/_2\varkappa k h r\sin 2(\theta - \delta) - M\cos\theta - N\sin\theta
\end{aligned} \tag{1.7}$$

where $h = h(t)$ are arbitrary functions and δ is a parameter.

The case $n = 2$. When $n = 2$, we take the formulas for the stresses and displacements (1.3) in the form

$$\begin{aligned}
&\sigma_r, \sigma_\theta = -{}^1/_2 f^{1/2} r^{-2}\{[(\psi'' - 8\psi)\chi]' + \nu\psi' \mp 12\psi'\chi\} \\
&\tau_{r\theta} = -f^{1/2} r^{-2}(\psi'' - 8\psi)\chi, \ \chi = [(\psi'' - 8\psi)^2 + 36\psi'^2]^{-1/4} \\
&u = -kfr^{-3}\psi', \ v = -2kfr^{-3}\psi
\end{aligned} \tag{1.8}$$

where $\psi(\theta)$ is given by the equation

$$[(\psi'' - 8\psi)\chi]'' + 12(\psi'\chi)' + \nu(\psi'' + 4\psi) = 0 \tag{1.9}$$

and the function $f(t)$ is found from the quadrature (1.6) for $m = {}^3/_2$. The graph of this function for $f_0 = 0$ is given in Fig.1, the solid and dashed lines corresponding to $c = 10$ and $c = 15$ respectively.

1°. Let us consider an infinite wedge with angle 2α, bent by the bending moment $M(t)$ applied to the vertex. Eq.(1.9) can be reduced to a system of first-order differential equations with boundary conditions

$$\psi' = s, \ s' = 8\psi + (\sqrt{2}/2)\tau R \tag{1.10}$$

$$\sigma' = -4\nu\psi, \quad \tau' = \sigma - \nu s - 12\sqrt{2}s/R$$
$$R = [\tau^2 + (\tau^4 + 144s^2)^{1/2}]^{1/2}$$
$$\theta = 0, \ \sigma = s = 0; \ \theta = \alpha, \ \sigma = \tau = 0$$

In order to establish the relations connecting $M(t)$ with $f(t)$, we shall consider the conditions of equilibrium of a sector of arbitrary radius r, with the centre at the vertex (Fig.2), which we imagine to be separated from the wedge. Equating to zero the sum of the moments of forces acting on this wedge-like body relative to the vertex, we have ($M = M(t)$)

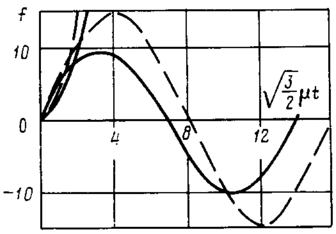

Fig.1

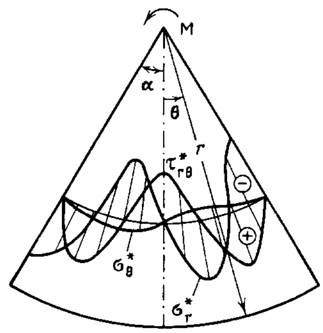

Fig.2

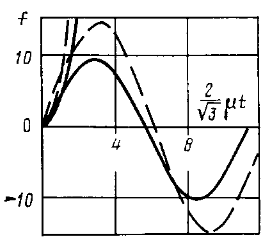

Fig.3

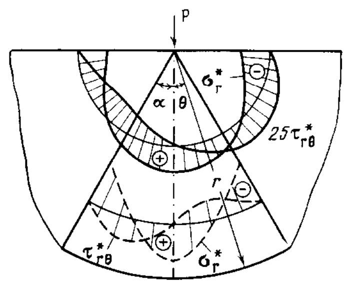

Fig.4

$$M + \int_{-\alpha}^{\alpha} \tau_{r\theta} r^2 \, d\theta = 0 \qquad (1.11)$$

Substituting the expressions $\tau_{r\theta}$ into (1.11), we obtain

$$M = Jf^{1/2}(t), \quad J = \int_{-\alpha}^{\alpha} \tau(\theta) \, d\theta$$

Finally we rewrite the formulas for the stresses and displacements (1.8) in the form

$$\sigma_r = -\frac{M}{2Jr^2}\left[\sigma - \frac{24\sqrt{2}\,s}{R}\right], \quad \sigma_\theta = -\frac{M}{2Jr^2}\sigma(\theta) \qquad (1.12)$$
$$\tau_{r\theta} = -\frac{M}{Jr^2}\tau(\theta), \quad u = -k\frac{M^2}{J^2}\frac{s(\theta)}{r^3}, \quad v = -2k\frac{M^2}{J^2}\frac{\psi(\theta)}{r^3}$$

Using the numerical solution of the boundary value problem (1.10) obtained on an ES-1023 computer by the trial-and-error method, we constructed the graphs (1 mm. corresponds to 0.4 units) for the relative stresses $\sigma^*_{ij} = -2Jr^2/M\ \sigma_{ij}$ with $\nu = 30$, $\alpha = \pi/6$ (Fig.2).

2°. When $M(t) = \text{const}$ and $2\alpha = \pi/2$, we use the particular solution $\psi = \cos 2\theta$ of (1.9) to obtain, from (1.8) and (1.11), the solution

$$\sigma_r = 2Mr^{-2}\sin 2\theta, \quad \tau_{r\theta} = -Mr^{-2}\cos 2\theta, \quad \sigma_\theta = 0 \qquad (1.13)$$
$$u = -{}^1/_6\, kM^2r^{-3}\sin 2\theta, \quad v = {}^1/_6\, kM^2r^{-3}\cos 2\theta$$

A numerical method of solving this problem for any α and n exists /9/.

The case $n = 3$. We write $n = 3$ in the relations (1.3)-(1.6). Eq.(1.4) reduces to the inhomogeneous, second-order differential equation

$$(\psi'' - 3\psi)\chi + \nu\psi = A \sin(\theta + \gamma) \qquad (1.14)$$
$$\chi = [(\psi'' - 3\psi)^2 + 16\psi'^2]^{-1/3}$$

where A and γ are arbitrary constants and in (1.6) $m = {}^4/_3$. Fig.3 shows graphs of $f(t)$ when $f_0 = 0$ (the notation is the same as in Fig.1).

The formulas for the stresses and displacements can be written as follows:

$$\sigma_r, \sigma_\theta = f^{1/3} r^{-1} [A \cos(\theta + \gamma) - (4(1 \pm 1)\psi'\chi] \qquad (1.15)$$
$$\tau_{r\theta} = f^{1/3} r^{-1} \tau, \quad \tau = A \sin(\theta + \gamma) - \nu\psi$$
$$u = -kfr^{-2}\psi', \; v = -kfr^{-2}\psi$$

The differential Eq.(1.14) can be reduced to a cubic equation in $(\psi'' - 3\psi)^{-1}$. Determining the real root of this equation, we obtain a differential equation which, when $A = 0$, reduces to a first-order differential equation. However, it is more convenient when a numerical solution is required, to reduce it to a system of two first-order equations

$$\psi' = (6\sqrt{3})^{-1} s, \quad s' = 18\sqrt{3}\psi - 4\sqrt{3}\nu\psi s / T \qquad (1.16)$$
$$T = T_+ - T_-, \quad T_\pm = [(s^2 + \nu^6\psi^6)^{1/2} \pm s]^{1/3}$$

1°. Let us consider an infinite wedge of angle 2α, with a concentrated axial force $P(t)$ applied to its vertex. Using the conditions of the problem, we shall have the following boundary conditions for the system of Eqs.(1.16):

$$\theta = 0, \; \psi = 0; \; \theta = \alpha, \; \psi = 0 \qquad (1.17)$$

Considering now the equilibrium of a sector of arbitrary radius r (Fig.4) which we imagine to be separated from the wedge, we obtain $(P = P(t))$

$$P + 2\int_0^\alpha (\sigma_r \cos\theta - \tau_{r\theta} \sin\theta) r\, d\theta = 0 \qquad (1.18)$$

After substituting the stress components into (1.18), we obtain

$$P = Jf^{1/3}(t), \quad J = \frac{4}{\sqrt{3}}\int_0^\alpha T \cos\theta\, d\theta - 2\nu\int_0^\alpha \psi \sin\theta\, d\theta \qquad (1.19)$$

Finally, the formulas for the stresses and displacements will have the form

$$\sigma_r = -\frac{2}{\sqrt{3}} \frac{P}{Jr} T(\psi, s), \quad \tau_{r\theta} = -\frac{\nu P}{Jr}\psi, \quad \sigma_\theta = 0 \qquad (1.20)$$
$$u = -\frac{k}{6\sqrt{3}} \frac{P^3}{J^3} \frac{s}{r^2}, \quad v = -k\frac{P^3}{J^3}\frac{\psi}{r^2}$$

Using the numerical solution of the boundary value problem (1.16)-(1.17), we have constructed the graphs of $\sigma_{ij}^* = -Jr/P\, \sigma_{ij}$ for $\nu = 60$ (Fig.4). The solid lines correspond to the case $\alpha = \pi/2$ (1 mm. corresponds to 4 units) and the dashed lines to $\alpha = \pi/6$ (1 mm. corresponds to 0.1 units).

2°. Let the infinite wedge be bent by a concentrated force $P(t)$ applied to its vertex in a direction perpendicular to the axis. In this antisymmetric case the boundary values for the system (1.16) are

$$\theta = 0, \; s = 0; \; \theta = \alpha, \; \psi = 0 \qquad (1.21)$$

Considering now the equilibrium of a sector of arbitrary radius r which we imagine to be separated from the wedge, we obtain

$$P + 2\int_0^\alpha (\sigma_r \sin\theta + \tau_{r\theta}\cos\theta)\, r\, d\theta = 0 \qquad (1.22)$$

Substituting the stress components into (1.22), we obtain a relation connecting $P(t)$ with $f(t)$ according to (1.19), while the formulas for the stresses will retain the form (1.20) and

$$J = \frac{4}{\sqrt{3}}\int_0^\alpha T \sin\theta\, d\theta + 2\nu\int_0^\alpha \psi\cos\theta\, d\theta$$

while the functions $\psi(\theta)$ and $s(\theta)$ will be found from the boundary value problem (1.16), (1.21).

The static problem of the compression of a wedge was solved for the general case of power-type hardening in /9, 10/. The dynamic problems for incompressible plastic media under the conditions of plane deformation were studied in /11, 12/.

2. Axisymmetric deformation. We shall write the relations of dynamic deformation of the medium in question, in spherical coordinates, as follows:

the equations of motion

$$\begin{aligned}
&\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{1}{r}(2\sigma_r-\sigma_\theta-\sigma_\varphi+\tau_{r\theta}\operatorname{ctg}\theta)=\rho\frac{\partial^2u}{\partial t^2}\\
&\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{1}{r}[(\sigma_\theta-\sigma_\varphi)\operatorname{ctg}\theta+3\tau_{r\theta}]=\rho\frac{\partial^2v}{\partial t^2}\\
&\frac{\partial\tau_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\varphi}}{\partial\theta}+\frac{1}{r}(2\tau_{\theta\varphi}\operatorname{ctg}\theta+3\tau_{r\varphi})=\rho\frac{\partial^2w}{\partial t^2}
\end{aligned}\tag{2.1}$$

the relations connecting the deformation and stress intensities and the relations connecting the deformation, displacement and stress components

$$\begin{aligned}
&\varepsilon_0=k\sigma_0^n\\
&\varepsilon_0=\sqrt{\frac{2}{3}}\,[(\varepsilon_r-\varepsilon_\theta)^2+(\varepsilon_\theta-\varepsilon_\varphi)^2+(\varepsilon_\varphi-\varepsilon_r)^2+\\
&\quad 6(\gamma_{r\theta}^2+\gamma_{\theta\varphi}^2+\gamma_{r\varphi}^2)]^{1/2}\\
&\sigma_0=\frac{1}{\sqrt{6}}[(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_\varphi)^2+(\sigma_\varphi-\sigma_r)^2+\\
&\quad 6(\tau_{r\theta}^2+\tau_{\theta\varphi}^2+\tau_{r\varphi}^2)]^{1/2}\\
&\varepsilon_r=\frac{\partial u}{\partial r},\quad \varepsilon_\theta=\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta},\quad \varepsilon_\varphi=\frac{u}{r}+\frac{v}{r}\operatorname{ctg}\theta\\
&2\gamma_{r\theta}=\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta},\quad 2\gamma_{\theta\varphi}=\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{w}{r}\operatorname{ctg}\theta\\
&2\gamma_{r\varphi}=\frac{\partial w}{\partial r}-\frac{w}{r},\quad \varepsilon_{ij}=\frac{1}{2}k\sigma_0^{n-1}(\sigma_{ij}-\delta_{ij}\sigma)
\end{aligned}\tag{2.2}$$

the stress and displacement components

$$\begin{aligned}
&\sigma_r=\sigma_\theta-2\varkappa f^{1/n}r^{-\lambda/n}[(2\lambda-3)\psi'+\lambda\psi\operatorname{ctg}\theta]\chi\\
&\sigma_\varphi=\sigma_\theta-2\varkappa(\lambda-3)f^{1/n}r^{-\lambda/n}(\psi'-\psi\operatorname{ctg}\theta)\chi\\
&\sigma_\theta=H+\rho k(2-\lambda)^{-1}f^{\cdot\cdot}r^{2-\lambda}(\psi''+\psi\operatorname{ctg}\theta)+\\
&\quad \varkappa n\lambda^{-1}f^{1/n}r^{-\lambda/n}\{([\psi''+(\psi\operatorname{ctg}\theta)'+\lambda(3-\lambda)\psi]\chi)'+\\
&\quad [\psi''+(\psi\operatorname{ctg}\theta)'+\lambda(3-\lambda)\psi]\chi\operatorname{ctg}\theta+\\
&\quad 6(1-\lambda)(\psi'+\psi\operatorname{ctg}\theta)\chi+2\lambda n^{-1}[(2\lambda-3)\psi'+\lambda\psi\operatorname{ctg}\theta]\chi\}\\
&\tau_{r\theta}=\varkappa f^{1/n}r^{-\lambda/n}[\psi''+(\psi\operatorname{ctg}\theta)'+\lambda(3-\lambda)\psi]\chi\\
&\tau_{r\varphi}=-\varkappa\lambda f^{1/n}r^{-\lambda/n}\varphi\chi\sin\theta,\quad \tau_{\theta\varphi}=\varkappa f^{1/n}r^{-\lambda/n}\varphi'\chi\sin\theta\\
&\chi=\{[\psi''+(\psi\operatorname{ctg}\theta)'+\lambda(3-\lambda)\psi]^2+4(\lambda^2-3\lambda+3)(\psi'^2+\\
&\quad \psi^2\operatorname{ctg}^2\theta)+4(\lambda^2-3)\psi'\psi\operatorname{ctg}\theta+(\varphi'^2+\lambda^2\varphi^2)\sin^2\theta\}^{(1-n)/2n}\\
&u=\varkappa kfr^{1-\lambda}(\psi'+\psi\operatorname{ctg}\theta),\quad v=\varkappa k(\lambda-3)fr^{1-\lambda}\psi\\
&w=\varkappa kfr^{1-\lambda}\varphi\sin\theta,\ \lambda\neq 0
\end{aligned}\tag{2.3}$$

Here $\varphi=\varphi(\theta)$ is an arbitrary function, and the remaining notation is unchanged. Expressions (2.3) represent a solution of the system (2.1)-(2.2), provided that $\psi(\theta)$ and $\varphi(\theta)$ satisfy the system of equations

$$\begin{aligned}
&([\psi''+(\psi\operatorname{ctg}\theta)'+\lambda(3-\lambda)\psi]\chi)''+([\psi''+(\psi\operatorname{ctg}\theta)'+\\
&\quad \lambda(3-\lambda)\psi]\chi\operatorname{ctg}\theta)'+\lambda n^{-1}(3-\lambda n^{-1})[\psi''+(\psi\operatorname{ctg}\theta)'+\\
&\quad \lambda(3-\lambda)\psi]\chi+2\lambda n^{-1}([(2\lambda-3)\psi'+\lambda\psi\operatorname{ctg}\theta]\chi)'+\\
&\quad 2\lambda n^{-1}(\lambda-3)(\psi'-\psi\operatorname{ctg}\theta)\chi\operatorname{ctg}\theta+6(1-\lambda)[(\psi'+\\
&\quad \psi\operatorname{ctg}\theta)\chi]'+\\
&\quad \lambda\nu n^{-1}(\lambda-2)^{-1}[\psi''+(\psi\operatorname{ctg}\theta)'+(\lambda-2)(\lambda-3)\psi]=0\\
&(\varphi'\chi\sin^3\theta)'+\lambda(\lambda n^{-1}-3)\varphi\chi\sin^3\theta+\nu\varphi\sin^3\theta=0
\end{aligned}\tag{2.4}$$

When studying the torsion of a solid of revolution, we should write in (2.3), (2.4) $\psi(\theta)=H(t)=0$.

When $\lambda=0$, the system of Eqs. (2.1), (2.2) admits of the solution ($g=g(t)$ is an arbitrary function)

$$\begin{aligned}
&\sigma_r=\sigma_\theta+\varkappa\sqrt{3}\,g^{1/n}\cos 2\theta,\quad \sigma_\varphi=\sigma_\theta-\varkappa\sqrt{3}\,g^{1/n}\sin^2\theta\\
&\sigma_\theta=H-\varkappa(\sqrt{3}/2)\,g^{1/n}\cos 2\theta-\varkappa(\sqrt{3}/48)\,\rho k g^{\cdot\cdot}r^2(1-\\
&\quad 3\cos^2\theta)\\
&\tau_{r\theta}=-\varkappa(\sqrt{3}/2)\,g^{1/n}\sin 2\theta,\quad \tau_{r\varphi}=\tau_{\theta\varphi}=w=0\\
&u=-\varkappa(\sqrt{3}/24)\,g(1-3\cos^2\theta),\quad v=-\varkappa(\sqrt{3}/16)\,gr\sin 2\theta
\end{aligned}\tag{2.5}$$

The case $n = 2$. Putting $n = 2$ in relations (2.3), (2.4) and writing $\varphi(\theta) = H(t) = 0$, we obtain the following expressions for the stress and displacement components:

$$\begin{aligned}
&\sigma_r = \sigma_\theta - 2\varkappa f^{1/2} r^{-2} (5\psi' + 4\psi \operatorname{ctg} \theta)\, \chi \\
&\sigma_\varphi = \sigma_\theta - 2\varkappa f^{1/2} r^{-2} (\psi' - \psi \operatorname{ctg} \theta)\, \chi \\
&\sigma_\theta = {}^1/_2 \varkappa f^{1/2} r^{-2} \{([\psi'' + (\psi \operatorname{ctg} \theta)' - 4\psi]\, \chi)' + \\
&\quad [\psi'' + (\psi \operatorname{ctg} \theta)' - 4\psi]\, \chi \operatorname{ctg} \theta + 2(\psi' - \psi \operatorname{ctg} \theta)\, \chi + \\
&\quad \nu (\psi' + \psi \operatorname{ctg} \theta)\} \\
&\tau_{r\theta} = \varkappa f^{1/2} r^{-2} [\psi'' + (\psi \operatorname{ctg} \theta)' - 4\psi]\, \chi \\
&\chi = \{[\psi'' + (\psi \operatorname{ctg} \theta)' - 4\psi]^2 + 28 (\psi'^2 + \psi^2 \operatorname{ctg}^2 \theta) + \\
&\quad 52 \psi' \psi \operatorname{ctg} \theta\}^{-1/4} \\
&u = \varkappa k f r^{-3} (\psi' + \psi \operatorname{ctg} \theta), \; v = \varkappa k f r^{-3} \psi
\end{aligned} \tag{2.6}$$

The function $f = f(t)$ is found according to the quadrature (1.6) with $m = {}^3/_2$, and $\psi(\theta)$ satisfies the equation

$$\begin{aligned}
&([\psi'' + (\psi \operatorname{ctg} \theta)' - 4\psi]\, \chi)'' + ([\psi'' + (\psi \operatorname{ctg} \theta)' - \\
&\quad 4\psi]\, \chi \operatorname{ctg} \theta)' + \\
&\quad 2\,[\psi'' + (\psi \operatorname{ctg} \theta)' - 4\psi]\, \chi + 2\,[(\psi' - \psi \operatorname{ctg} \theta)\, \chi]' + \\
&\quad 4\,(\psi' - \psi \operatorname{ctg} \theta)\, \chi \operatorname{ctg} \theta + {}^1/_2 \nu\, [\psi'' + (\psi \operatorname{ctg} \theta)' + 2\psi] = 0
\end{aligned} \tag{2.7}$$

1°. Let us consider an infinite cone of angle 2α, compressed by an axial force $P(t)$. Eq.(2.7) reduces to a system of four first-order differential equations

$$\begin{aligned}
&\psi' = s - \psi \operatorname{ctg} \theta, \; s' = 4\psi' + \tau\omega \\
&\tau' = \sigma - \nu s - \tau \operatorname{ctg} \theta - 2\omega^{-1} (s - 2\psi \operatorname{ctg} \theta) \\
&\sigma' = -2\tau - 2\nu\psi - 4\omega^{-1} (s - 2\psi \operatorname{ctg} \theta) \operatorname{ctg} \theta \\
&\omega = (\sqrt{2}/2)\, \{\tau^2 + [\tau^4 + 16\,(7s^2 - s\psi \operatorname{ctg} \theta + \psi^2 \operatorname{ctg}^2 \theta)]^{1/2}\}^{1/2}
\end{aligned} \tag{2.8}$$

The boundary conditions for system (2.8) are

$$\theta = 0, \; \tau = \psi = 0; \; \theta = \alpha, \; \sigma = \tau = 0 \tag{2.9}$$

To establish a relation connecting $P(t)$ and $f(t)$, we will consider a cone-like body bounded by an arbitrary spherical surface of radius r, with centre at the vertex (Fig.5), which we imagine to be separated from the rest. We have

$$P + 2\pi \int_0^\alpha (\sigma_r \cos\theta - \tau_{r\theta} \sin\theta)\, r^2 \sin\theta\, d\theta = 0 \tag{2.10}$$

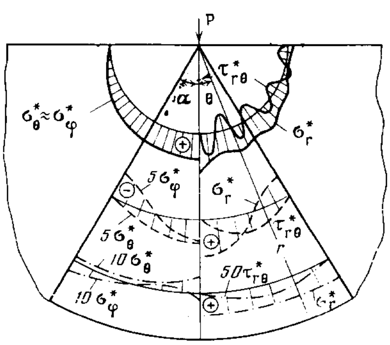

Fig.5

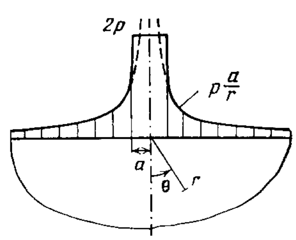

Fig.6

Substituting into (2.10) expressions for σ_r and $\tau_{r\theta}$ written in terms of ψ, s, τ and σ, we obtain

$$\begin{aligned}
&P = \pi J f^{1/2} \\
&J = J(\alpha) = \int_0^\alpha \left[\sigma - \frac{4}{\omega}(5s - \psi \operatorname{ctg} \theta)\right] \cos\theta \sin\theta\, d\theta - 2\int_0^\alpha \tau \sin^2\theta\, d\theta
\end{aligned} \tag{2.11}$$

Finally, we obtain the following expressions for the stresses and displacements:

$$\sigma_r = -\frac{P}{2\pi J r^2}\left[\sigma - \frac{4}{\omega}(5s - \psi \operatorname{ctg}\theta)\right], \quad \sigma_\theta = -\frac{P\sigma(\theta)}{2\pi J r^2} \tag{2.12}$$
$$\sigma_\varphi = -\frac{P}{2\pi J r^2}\left[\sigma - \frac{4}{\omega}(s - 2\psi \operatorname{ctg}\theta)\right], \quad \tau_{r\theta} = -\frac{P}{\pi J r^2}\tau(\theta)$$
$$u = -k\frac{P^3}{\pi^3 J^3}\frac{s(\theta)}{r^3}, \quad v = -k\frac{P^3}{\pi^3 J^3}\frac{\psi(\theta)}{r^3}$$

Using the numerical solution of the boundary value problem (2.8), (2.9), we give in Fig.5 graphs of $\sigma_{ij}^* = -2\pi J r^2 / P\sigma_{ij}$ for $\nu = 30$. The solid lines correspond to $\alpha = \pi/2$ (1 mm corresponds to 10 units) and the dashed lines to $\alpha = \pi/6$ (1 mm corresponds to 0.5 units).

2°. Let a concentrated force $P(t)$ be applied at the point of an infinite body regarded as the origin of coordinates, along the axis $\theta = 0$. In this case the system of Eqs.(2.8) should be integrated using the boundary conditions $\theta = 0, \pi,\ \tau = \psi = 0$. The condition of equilibrium of a mentally separated sphere of arbitrary radius r and the centre situated at the origin of coordinates is written in the form (2.10) with $\alpha = \pi$. The formulas for the stresses and displacements and the dependence of P on $f(t)$, are found according to (2.11), (2.12), with $J = J(\pi)$.

The case $n = 3$. When $n = 3$, the formulas for the stresses and displacements (2.3) without torsional deformations, can be written in the form

$$\sigma_r, \sigma_\theta = -f^{1/3}r^{-1}\left[\left(\frac{s'}{\omega}\right)' + \frac{s'}{\omega}\operatorname{ctg}\theta - (9 \pm 3)\frac{s}{\omega} - \nu s\right], \quad \sigma_\varphi = \sigma_\theta \tag{2.13}$$
$$\tau_{r\theta} = -f^{1/3}r^{-1}\frac{s'}{\omega}, \quad \omega = (s'^2 + 12s^2)^{1/3}, \quad u = -kf\frac{s}{r^2}, \quad v = 0$$

where s satisfies the equation

$$\left(\frac{s'}{\omega}\right)'' + \left(\frac{s'}{\omega}\operatorname{ctg}\theta\right)' + 2\frac{s'}{\omega} - 6\left(\frac{s}{\omega}\right)' - \nu s' = 0 \tag{2.14}$$

and $f(t)$ is determined according to (1.6) with $m = {}^4/_3$.

Let us consider a half-space whose surface is acted upon by the loads (Fig.6)

$$\theta = \frac{\pi}{2}; \quad \sigma_\theta = -p\frac{a}{r}\cos\beta, \quad \tau_{r\theta} = p\frac{a}{r}\sin\beta, \quad p = \frac{1}{a}f^{1/3}(t) \tag{2.15}$$

where a and β are given parameters. Introducing the notation $s' = \tau\omega$ we obtain, from the expression for ω in (2.13), a cubic equation in $1/s'$. Having determined the real root, we can reduce Eq.(2.14) to a system of three first-order ordinary differential equations

$$s' = 6\tau s/\Omega, \ \tau' = \sigma + \nu s - \tau\operatorname{ctg}\theta + \Omega, \ \sigma' = -2\tau$$
$$\Omega = \Omega_+ - \Omega_-, \quad \Omega_\pm = [(\tau^6 + 81s^2)^{1/2} \pm 9s]^{1/3}$$

with boundary conditions

$$\theta = 0, \ \tau = 0; \ \theta = \pi/2, \ \sigma = \cos\beta, \ \tau = -\sin\beta$$

The formulas for the stresses and displacements can finally be written in the form

$$\sigma_r = -p(a/r)(\sigma - \Omega), \ \sigma_\theta = -p(a/r)\sigma(\theta), \ \sigma_\varphi = \sigma_\theta$$
$$\tau_{r\theta} = -p(a/r)\tau(\theta), \ u/a = kp^3(a/r)^2 s(\theta), \ v = 0$$

When there are no tangential loads on the surface $\theta = \pi/2$, we should write $\beta = 0$.

3. The static case ($\nu = 0$). Let us consider the static deformation of bodies with power-type hardening $\varepsilon_0 = k\sigma_0^n$.

1°. Let a uniformly distributed pressure p be applied to one of the infinite edges of a quarter-plane. Putting in (1.7) $g(t) = \text{const}$ $M(t) = \text{const}$ and using the boundary conditions $\sigma_\theta = -p$ when $\theta = 0$ and $\sigma_\theta = 0$ when $\theta = \pi/2$, we finally obtain

$$\sigma_r, \ \sigma_\theta = -{}^1/_2 p\,(1 \mp \cos 2\theta), \ \tau_{r\theta} = -{}^1/_2 p\sin 2\theta$$
$$u = {}^1/_2 k\,({}^1/_2 p)^n r\cos 2\theta - M\sin(\theta + \delta)$$
$$v = -{}^1/_2 k\,({}^1/_2 p)^n r\sin 2\theta - M\cos(\theta + \delta)$$

A numerical method exists /9/ for solving this problem for any value of the wedge angle.

2°. Let us consider an infinite cone with angle 2α, compressed by an axial force P applied at its vertex (generalization of the Boussinesq solution to the case of an incompressible material). Assuming that $\nu = 0$, $\lambda = 2n$ and $\varphi(\theta) = 0$, we can reduce Eq.(2.4) to a system of four first-order differential equations

$$\psi' = s - \psi \operatorname{ctg}\theta,\ s' = 2n(2n-3)\psi + \tau\omega$$
$$\tau' = \sigma - \tau\operatorname{ctg}\theta - {}^1/_2 S,\ \sigma' = -2\tau - S\operatorname{ctg}\theta \qquad S = 4(2n-3)\,\omega^{-1}(s - 2\psi\operatorname{ctg}\theta) \tag{3.1}$$

and ω is found from the power equation

$$\omega^{2n/(n-1)} - \tau^2\omega^2 - 4(4n^2 - 6n + 3)s^2 + 4(2n-3)^2(s - \psi\operatorname{ctg}\theta)\psi\operatorname{ctg}\theta = 0$$

Using the corresponding expressions for the stresses and displacements (2.3), we arrive at the boundary conditions for system (3.1)

$$\theta = 0,\ \tau = \psi = 0;\ \theta = \alpha,\ \sigma = \tau = 0$$

When $\nu = 0$, Eq.(2.10) yields

$$P = \pi J f^{1/n}, \quad J = J(\alpha, n) = \int_0^\alpha \Big[\sigma - \frac{4n-3}{2n-3}S - \frac{12}{\omega}(2n-1)\psi\operatorname{ctg}\theta\Big]\cos\theta\sin\theta\,d\theta - 2\int_0^\alpha \tau\sin^2\theta\,d\theta$$

The formulas for the stresses and displacements are

$$\sigma_r = -\frac{P}{2\pi J r^2}\Big[\sigma - \frac{4n-3}{2n-3}S - \frac{12}{\omega}(2n-1)\psi\operatorname{ctg}\theta\Big]$$
$$\sigma_\theta = -\frac{P}{2\pi J r^2}\sigma(\theta), \quad \sigma_\varphi = -\frac{P}{2\pi J r^2}(\sigma - S), \quad \tau_{r\theta} = -\frac{P}{\pi J r^2}\tau(\theta)$$
$$u = -k\frac{P^n}{\pi^n J^n}\frac{s(\theta)}{r^{2n-1}}, \quad v = -(2n-3)k\frac{P^n}{\pi^n J^n}\frac{\psi(\theta)}{r^{2n-1}} \tag{3.2}$$

The results of a numerical solution for $n = 2,\ \alpha = \pi/6,\ \nu = 0$ are shown in Fig.5 by the dot-dash lines (1 mm corresponds to 2 units).

When $n = 1$, we put $\psi = -\sin\theta$ and $\sigma = \tau = 0$ to obtain, from (3.2), the corresponding formulas for the linearly elastic material.

The stress state of a half-space with power-type hardening of the material caused by the application of a concentrated normal force, was considered in /13/ where the author had succeeded in determining the displacement of a point on the surface of the half-space, apart from a constant factor.

3°. Let a concentrated force P be applied at the point of an infinite body corresponding to the origin of coordinates along the axis $\theta = 0$. In this case the system of Eqs.(3.1) should be integrated with the boundary conditions $\theta = 0, \pi,\ \tau = \psi = 0$. The formulas for the stresses and displacements are given by (3.2), where we must assume that $J = J(\pi, n)$.

4°. Let us consider a cone whose side surface is acted upon by uniformly distributed normal and tangential forces. Using (2.5) and satisfying the boundary conditions $\sigma_\theta = -p,\ \tau_{r\theta} = q$, we obtain, when $\theta = \alpha$,

$$\sigma_r,\ \sigma_\theta = -p - q(\cos 2\alpha \pm \cos 2\theta)/\sin 2\alpha$$
$$\sigma_\varphi = -p + q\operatorname{tg}\alpha, \quad \tau_{r\theta} = q\sin 2\theta/\sin 2\alpha$$
$$u = \frac{\sqrt{3}}{24}k\left(\frac{2q}{\sqrt{3}}\right)^n\frac{r(3\cos^2\theta - 1)}{\sin^n 2\alpha}, \quad v = -\frac{\sqrt{3}}{16}k\left(\frac{2q}{\sqrt{3}}\right)^n\frac{r\sin 2\theta}{\sin^n 2\alpha}$$

5°. Let distributed loads be applied at the surface $\theta = \pi/2$ of the half-space (Fig.6) according to the law

$$\sigma_\theta = -p(a/r)^m\cos\beta, \quad \tau_{r\theta} = p(a/r)^m\sin\beta, \quad 0 < m < 2$$

The case $m = 1$ is shown in Fig.6.

Assuming that $\nu = 0,\ \lambda = mn$ and $\varphi(\theta) = 0$, we reduce Eq.(2.4) to the following system of four first-order differential equtions:

$$\psi' = s - \psi\operatorname{ctg}\theta, \quad s' = mn(mn-3)\psi + \tau\omega$$
$$\tau' = \sigma - \tau\operatorname{ctg}\theta - 2\omega^{-1}\{[m(2mn-3) + 3(1-mn)]s - m(mn-3)\psi\operatorname{ctg}\theta\}$$
$$\sigma' = m(m-3)\tau - 2m\omega^{-1}(mn-3)(s - 2\psi\operatorname{ctg}\theta)\operatorname{ctg}\theta \tag{3.3}$$

where ω is given by the power equation

$$\omega^{2n/(n-1)} - \tau^2\omega^2 - 4(m^2n^2 - 3mn + 3)s^2 + 4(mn-3)^2(s - \psi\operatorname{ctg}\theta)\psi\operatorname{ctg}\theta = 0$$

The boundary conditions are

$$\theta = 0,\ \tau = \psi = 0;\ \theta = \pi/2,\ \sigma = m\cos\beta,\ \tau = -\sin\beta$$

The formulas for the stresses and displacements are

$$\sigma_r = -\frac{p}{m}\left(\frac{a}{r}\right)^m \left[\sigma - \frac{m(2mn-3)}{2(2n-3)} S - \frac{6m(mn-1)}{\omega}\psi \operatorname{ctg}\theta\right]$$

$$\sigma_\varphi = -\frac{p}{m}\left(\frac{a}{r}\right)^m \left[\sigma - \frac{m(mn-3)}{2(2n-3)} S\right], \quad \sigma_\theta = -\frac{p}{m}\left(\frac{a}{r}\right)^m \sigma(\theta)$$

$$\tau_{r\theta} = -p\left(\frac{a}{r}\right)^m \tau(\theta), \quad \frac{u}{a} = -kp^n\left(\frac{a}{r}\right)^{mn-1} s(\theta)$$

$$\frac{v}{a} = -(mn-3)\,kp^n\left(\frac{a}{r}\right)^{mn-1}\psi(\theta)$$

When only normal forces are applied to the surface $\theta = \pi/2$, we must write $\beta = 0$.

Just as in /14/, we can consider theproblem of the dynamic deformation of component bodies in contact with each other through the coordinate surfaces, with different deformation moduli k and the same degree of hardening n.

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